



IIT ROORKEE



NPTEL ONLINE
CERTIFICATION COURSE

QUANTITATIVE INVESTMENT MANAGEMENT

LECTURE 44

Option Pricing – Binomial Model – 2

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INTERPRETATION OF q_u AND q_d AS PROBABILITIES

- It has been shown above that
- $q_u + q_d = 1$; $0 < q_u, q_d < 1$
- We can, therefore, interpret q_u, q_d as some probabilities. we call them synthetic or q-probabilities
- Please note that they are purely mathematical constructs and are, in no way, related to real world probabilities. They are simply certain mathematical quantities that seem to satisfy the laws of probability.



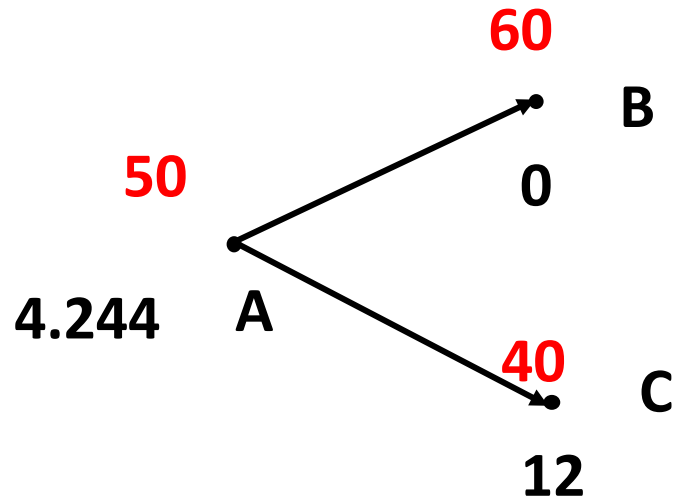
SINGLE PERIOD BINOMIAL MODEL

EXAMPLE 1

- Consider a 1-year European put with a strike price of 52 on a stock whose current price is 50. In the single time step (of one year) the stock price either moves up by 20% or moves down by 20%. Let the risk-free interest rate be 5%. Calculate the current price of the option using a single step binomial model.



ONE STEP MODEL: EUROPEAN PUT



$$q_u = \left(\frac{S_0 e^{rT} - S_d}{S_u - S_d} \right) = \left(\frac{50e^{0.05 \times 1} - 40}{60 - 40} \right) = 0.6282$$

$$q_d = \left(\frac{S_u - S_0 e^{rT}}{S_u - S_d} \right) = 0.3718$$

$$f_0 = e^{-rT} (q_u f_u + q_d f_d) = e^{-0.05 \times 1} (0.3718 \times 12) = 4.244$$

$u = 1.20$; $d = 0.80$; $T = 1$ year; $r = 5\%$

Risk neutral probabilities

$$q_u = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \times 1} - 0.80}{1.20 - 0.80} = 0.6282$$

Stock prices at various nodes

$$S_u = 60; S_d = 40; S_0 = 50$$

Claim values at various nodes : $f(S_T) = \max(K - S_T, 0)$

$$f_d = \max(52 - 40, 0) = 12; f_u = \max(52 - 60, 0) = 0$$

RISK NEUTRAL VALUATION: $E_Q(S_T)$

Since we have identified q_u, q_d as probabilities, we can also define expectation of stock price as $E_Q(S_t) =$

$$S_d \left(\frac{S_u - S_0 e^{rT}}{S_u - S_d} \right) + S_u \left(\frac{S_0 e^{rT} - S_d}{S_u - S_d} \right) = S_0 e^{rT}$$



IMPLICATIONS: RISK NEUTRAL PRICING

- Thus, the expected return on stock price (calculated with reference to q-probabilities) equals the riskfree rate.
- Now, if the stock price (**a risky asset**) is giving an expected return of the riskless asset, then there are two issues:
 - **All investors are accepting the same return irrespective of the risk profile of the asset;**
 - **That return is the riskfree return.**



RISK NEUTRALITY OF INVESTORS

- All the investors do not care at all about risk i.e. each investor is completely risk neutral or indifferent to risk.
- It is only then that the same return will be acceptable to them even on risky assets.



RISKFREE RETURN

- The fact that in a risk neutral world, only riskfree return is possible and no higher return follows from arbitrage considerations. If any other asset X yields a higher expected return even if it is more risky, all investors will invest in X because, by assumption, they are indifferent to risk whence return is the only consideration for investment.
- This will enhance the demand and price of X , thereby reducing its return to the riskfree rate at equilibrium.

- **The bottomline of the above analysis is that the q -probabilities reflect the probabilities of stock price movements in a risk neutral world i.e. in a world where risk has no significance to each and every investor and return is the only investment criterion. Hence, these probabilities are also called risk neutral probabilities.**



IMPORTANT

- Thus, the price of a derivative is the present value of the expected payoff from the derivative on maturity, the expectation being calculated with reference to risk neutral probabilities.

STOCK'S REAL WORLD EXPECTED RETURN IRRELEVANT: WHY?

- We claim that the stock's real world probabilities are irrelevant to the pricing of its derivative.**
- Now, the expected stock price at T depends on the probability of upswing/ downswing at T. Hence, the real life probabilities of stock price upswing and downswing are already captured in the stock price.**



- Our model assumes the construction of a completely riskfree asset by a combination of the derivative and the underlying.
- We are pricing the derivative by pricing this riskfree portfolio (that gives riskfree return) and the stock.
- The real world probabilities are already captured by the stock price.
- Hence, the real world probabilities do not come into play in derivative's pricing.

NO ARBITRAGE VS RISK NEUTRAL PRICING

- In the no-arbitrage approach, we set up a riskless portfolio consisting of a position in the option and a position in the stock. By setting the return on the portfolio equal to the risk-free interest rate, we are able to value the option.
- When we use risk-neutral valuation, we first choose probabilities for the branches of the tree so that the expected return on the stock equals the risk-free interest rate. We then value the option by calculating its expected payoff and discounting this expected payoff at the risk-free interest rate.

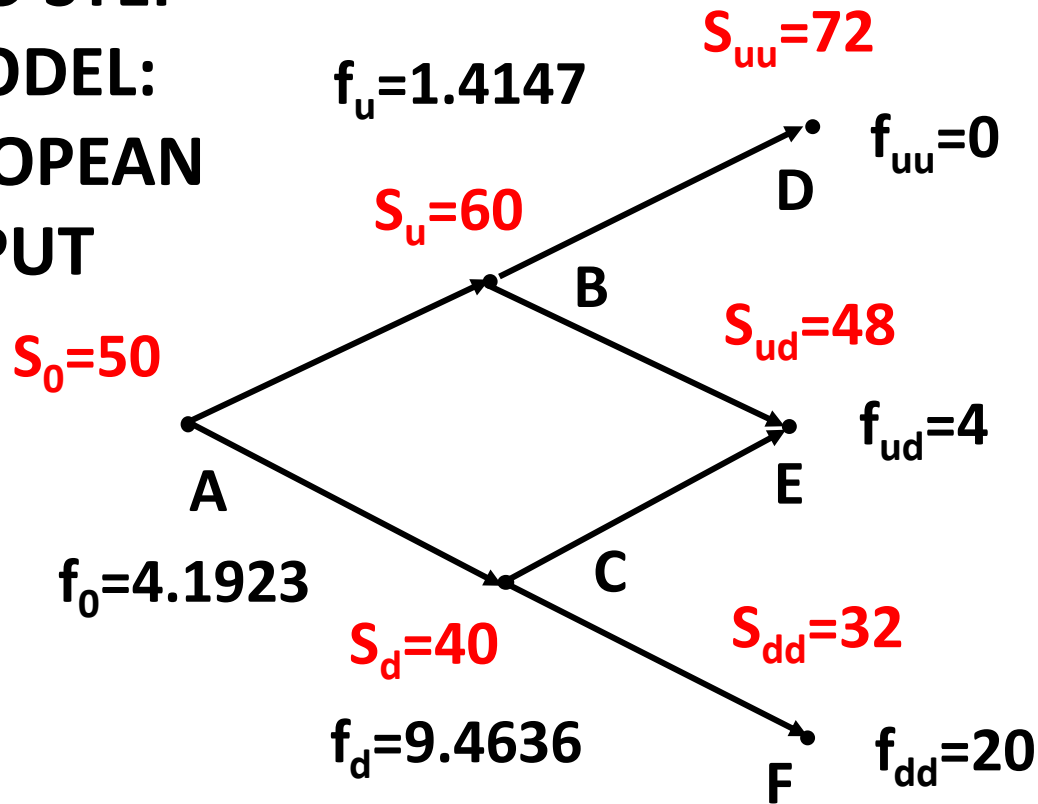
TWO PERIOD BINOMIAL MODEL

EXAMPLE 1

- Consider a 2-year European put with a strike price of 52 on a stock whose current price is 50. In each time step (of one year) the stock price either moves up by 20% or moves down by 20%. Let the risk-free interest rate be 5%. Calculate the current price of the option using a two step binomial model.



TWO STEP MODEL: EUROPEAN PUT



$u = 1.20; d = 0.80; T = 1 \text{ year}; r = 5\%$

Risk neutral probabilities

$$q_u = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \times 1} - 0.80}{1.20 - 0.80} = 0.6282$$

Stock prices at various nodes

$S_{uu} = 72; S_{ud} = S_{du} = 48; S_{dd} = 32; S_u = 60; S_d = 40; S_0 = 50$

Claim values at various nodes : $f(S_T) = \max(K - S_T, 0)$

$$f_{uu} = \max(52 - 72, 0) = 0; f_{ud} = \max(52 - 48, 0) = 4$$

$$f_{dd} = \max(52 - 32, 0) = 20$$

$$f_u = e^{-rT} [qf_{uu} + (1-q)f_{ud}]$$

$$= 0.9512 (0.6282 \times 0 + 0.3718 \times 4) = 1.4147$$

$$f_d = e^{-rT} [qf_{ud} + (1-q)f_{dd}]$$

$$= 0.9512 (0.6282 \times 4 + 0.3718 \times 20) = 9.4636$$

$$f_0 = e^{-rT} [qf_u + (1-q)f_d]$$

$$= 0.9512 (0.6282 \times 1.4147 + 0.3718 \times 9.4636)$$

$$= 4.1923$$